Official Insignia for the 2018 Falcon Heavy Test Flight (SpaceX, 2018)

Rocket Launch Trajectory Analysis Exploration

A MATHEMATICAL STUDY INTO THE TRAJECTORY OF THE FALCON HEAVY TEST FLIGHT MISSION OF 2018

HL Mathematics Exploration IA | 05/06/2020

RATIONALE

I have always been interested in space travel. The prospect of exploring new planets excites me entirely. Last year was the moon landing's 50th anniversary. This was when I first deeply considered my interest and began following the progress and activity of SpaceX, tracking their progress in reusable rockets and innovative launch procedures. More recently, I watched the film, *Hidden Figures*. I became intrigued by rocket trajectory analysis and began to look for an opportunity to perform these mathematical trajectory calculations.

INTRODUCTION

My most memorable experience following SpaceX was the 2018 Falcon Heavy Test Flight, still the world's most powerful rocket with 64 metric tonnes in orbital lift capacity (SpaceX, 2020). Thus, it is the best candidate for my trajectory study. Luckily, SpaceX streams telemetry data for their flights live, with predictive figures prior to launch. However, SpaceX do not publicly release analysed data, meaning mathematical interpolation and extrapolation of this raw telemetry is generally the role of the space exploration fan community, using their own methods. In this exploration, I will contribute to this community, providing my own mathematical interpretation of the telemetry data and assessing its accuracy against other analyses. My piecewise cartesian method, with a foundation in regression analysis, will see itself more accessible than some of the more complex processes used in the community. Additionally, with my prior interest, I will pair my Falcon Heavy knowledge with these mathematical concepts to generate the most accurate and visual results possible.

The 2018 Falcon Heavy Test Flight's mission objective was to launch the payload, Elon Musk's personal Tesla Roadster, into heliocentric orbit with maximum altitude equivalent to Mars' orbital radius (SpaceX, 2018). To accomplish this, the Falcon Heavy used its two stages: a vertical launch, before gradually turning until the rocket's path was tangential to the Earth's curvature. This was at the payload's highest Earth altitude before moving outwards towards Mars. The full launch curve is shown in the flight path found in Figure 1, while I only focus on the main curved path of the payload.

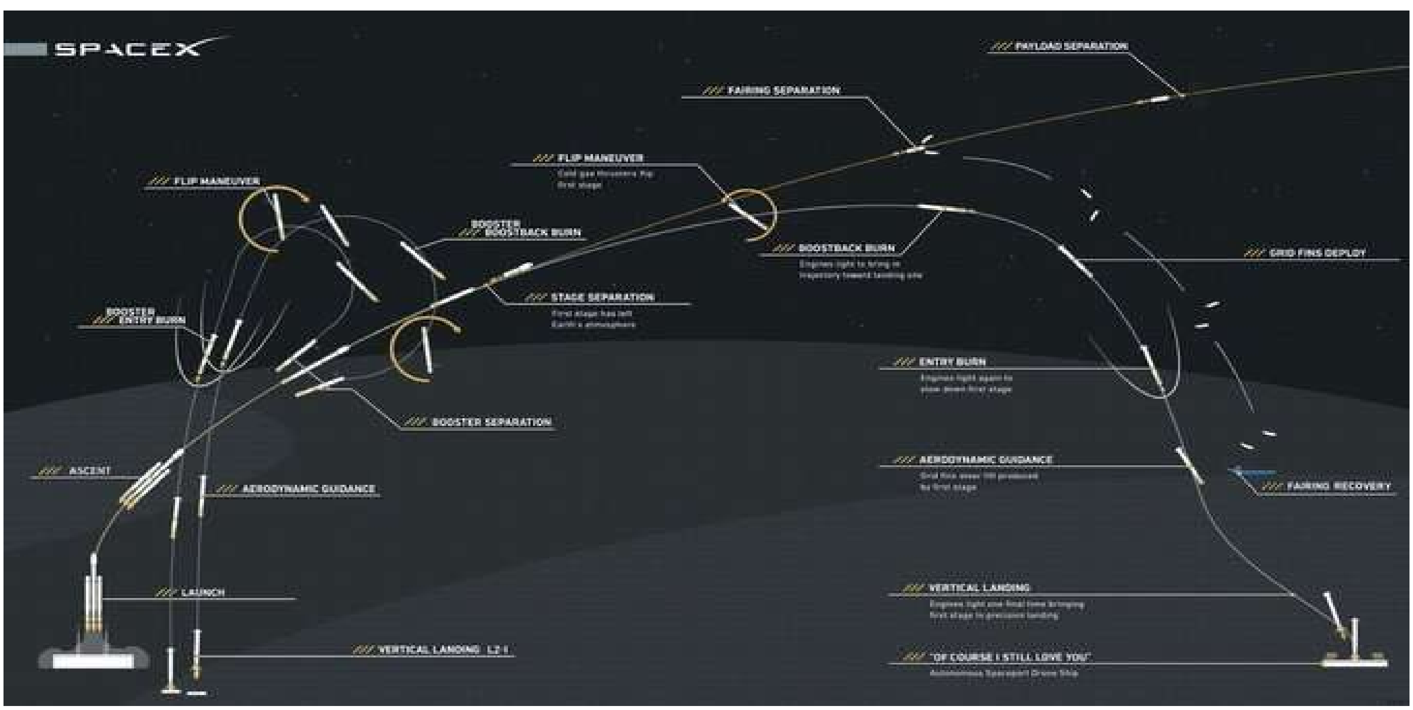

Figure 1: Flight path for the Falcon Heavy Test Flight (Musk, 2018)

DATA IDENTIFICATION AND INITIAL REGRESSION ANALYSIS

Alongside SpaceX's live webcast of the test flight was the live telemetry data. This showed time after launch, current altitude from Earth's surface and velocity magnitude. The figures were accurate to one second, so I manually created a table with time, altitude and velocity values every second for the first 500 seconds after launch. After this time, the altitude began to decline and I was only interested in the ascension of the rocket (SpaceX, 2018). Figure 2 provides the first 19 seconds of data, after I standardised the values into SI units: seconds [s], metres [m] and metres per second [m·s^{-1}] respectively. I left the same number of significant figures as in the original telemetry data.

Time (s)	Altitude (m)	Velocity (m·s^{-1})	Time (s)	Altitude (m)	Velocity (m·s^{-1})
0	0	0.000	**10**	172	42.089
1	1	1.056	**11**	217	47.785
2	3	2.872	**12**	267	53.063
3	7	6.111	**13**	323	58.341
4	15	10.143	**14**	384	64.729
5	28	15.422	**15**	452	71.115
6	46	20.837	**16**	526	76.948
7	70	26.115	**17**	606	82.785
8	98	31.392	**18**	692	89.170
9	132	36.674	**19**	784	95.145

Figure 2: Raw time, altitude, and velocity values for first 19 seconds of ascension

Next, I plotted the two graphs: altitude against time (Figure 3) and velocity against time (Figure 4).

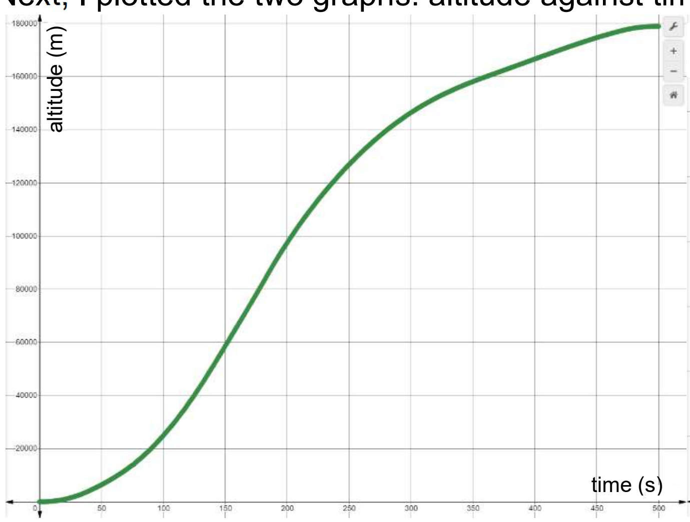

Figure 3: Raw data for altitude vs. time during ascension

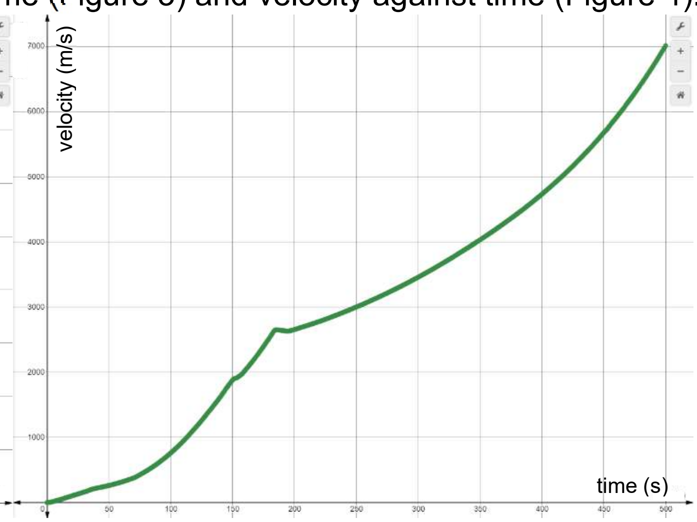

Figure 4: Raw data for velocity vs. time during ascension

To perform cartesian analysis, I chose a function to represent the data. This meant a regression analysis, in using a function to represent a data trend was necessary. The smooth transition from minimum to maximum with one inflection point in Figure 3 reminded me of the distinctive shape of a logistic function. These have the form $y = \frac{L}{1+e^{-k(x-c)}}$ where L is an asymptotic maximum of y, c is the x position of an inflection point and k relates to steepness at the inflection point (New York University | Center for Neural Science, 2020). I made L the maximum altitude reached $(178967m)$ and it looked as though the inflection point was at $x \approx 190s$, being my c value. I then used Desmos, a graph manipulation tool, to compare this function with the altitude data. I manually changed k until my logistic regression seemed to fit best. The coefficient of determination (R^2 value) for the curve when $k = 0.02$ was $R^2 \approx 0.9838$. This is a very strong correlation, as shown in Figure 5.

Although the correlation is strong, I was unsatisfied with how the regression did not capture the data's y-intercept value of 0 or its abnormal curvature after the inflection point. To capture these data trend changes better, I split the raw data into subsections. I thought if each subsection had its own accurate regression, my regression would be better over the total 500 seconds. I made the splits at the key launch stages in Figure 1: booster engine cut off (BECO) $(t \approx 153)$; main engine cut off (MECO) $(t \approx 186)$ and; second stage beginning $(t \approx 196)$. SpaceX published these times prior to launch and I chose them as they mark changes in the forces being applied to the payload from thrusters (SpaceX, 2018).

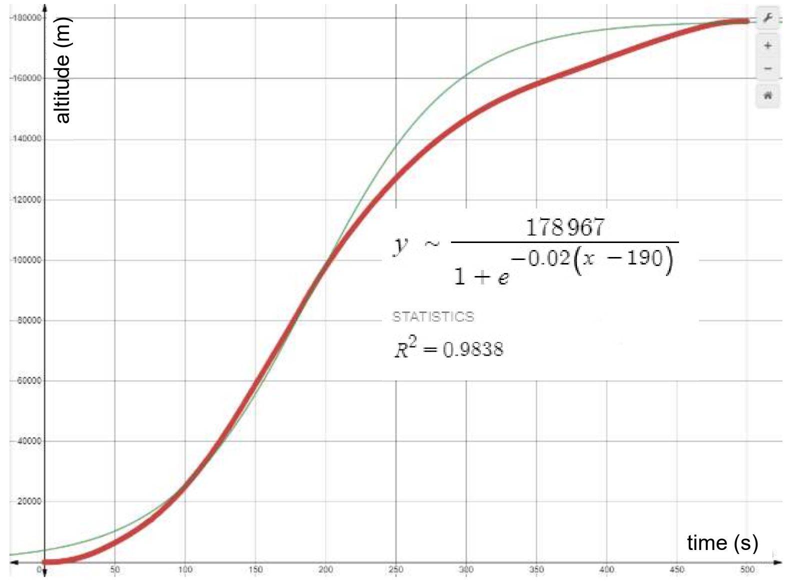

Figure 5: Curve fitting of logistic function (green) to altitude vs. time data (red) (Desmos, Inc., 2020)

For the most accurate and systematic analysis, I used polynomial regression analysis. I instructed Desmos to generate a polynomial in the form: $y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$; to each data subsection, knowing that a higher degree (n value) would give more accuracy in the domain (Agarwal, 2018). Desmos then provided me with the parameters, $a_n, a_{n-1}, \ldots, a_0$, for the n^{th} degree polynomial with sufficient accuracy to warrant a displayed $R^2 \approx 1$. The real R^2, however, would be slightly lower. When limited to their respective domains, the polynomials formed Figure 6 and Figure 7. For the regression of the first subsection in each graph, I set the constant, a_0, to 0. Since both altitude and velocity began at 0, I wanted to emulate this by removing vertical shift.

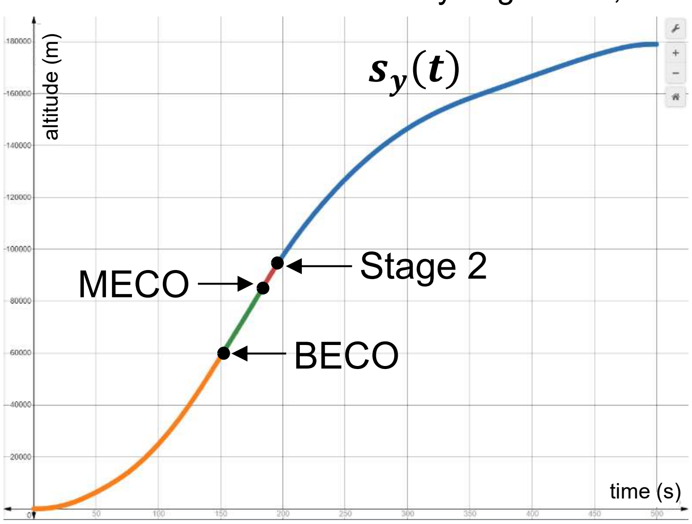

Figure 6: Piecewise-defined polynomial regression analysis of altitude vs. time (Desmos, Inc., 2020)

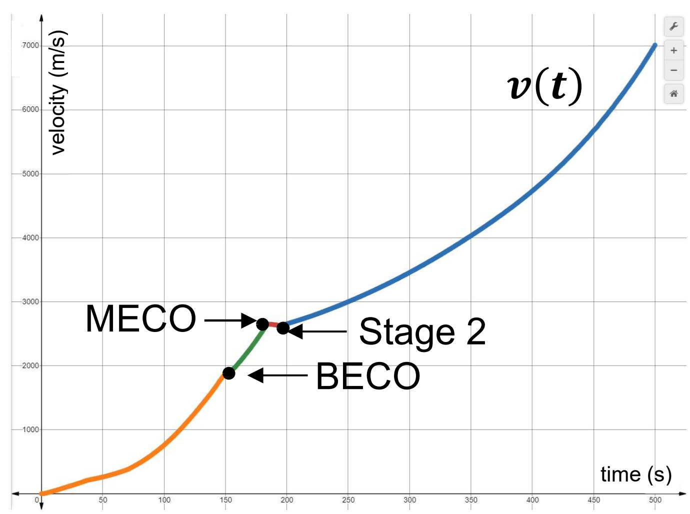

Figure 7: Piecewise-defined polynomial regression analysis of velocity vs. time (Desmos, Inc., 2020)

I was extremely pleased with the much better trend and $R^2 \approx 1$ values from data splitting. This was also important as it segregated the distinctively changing features of the velocity graph. These are now piecewise functions, defined differently for different domain intervals (Lumen Learning, 2020). This was also much more accurate than my 'guess and check' for finding the logistic k value.

Desmos cited some of my coefficients, generally for the polynomial degrees above 10, as 0. I thus thought they were unnecessary. However, I was shocked that when I removed these terms, the function changed drastically. I believe that these parameters were not 0 but were so small that Demos could not show them. So, I was unable to provide the exact polynomial equations here. Instead, I defined Figure 6 as $s_y(t)$, being the payload's y directional displacement with respect to time, and Figure 7 as $v(t)$, being its instantaneous velocity magnitude with respect to time.

VARIABLE DIRECTION DEFINITIONS

After referring to the y direction, I will define my two directions of movement: the x and y direction. Since SpaceX's altitude data is relative to Earth's position, it would be unreasonable at this stage, to assign an unmoving 'cosmic' linear y or x direction. As such, I define a coordinate system relating to the Earth's surface, such that: at any one position in space:

1. The direction of the increasing y coordinate is radially away from the centre of the Earth.
2. The direction of the increasing x coordinate is tangential to Earth's curvature directly below but points away from the launch site.

These directions, for further use, are illustrated in Figure 8. However, they rely on the assumptions that: firstly, the Falcon Heavy does not travel at least halfway around the Earth (which it did not) and; secondly, the Falcon heavy will not turn into a third axis, mapping its motion on a two-dimensional plane (which was almost always the case) (SpaceX, 2018). Hence, these assumptions seem reasonable.

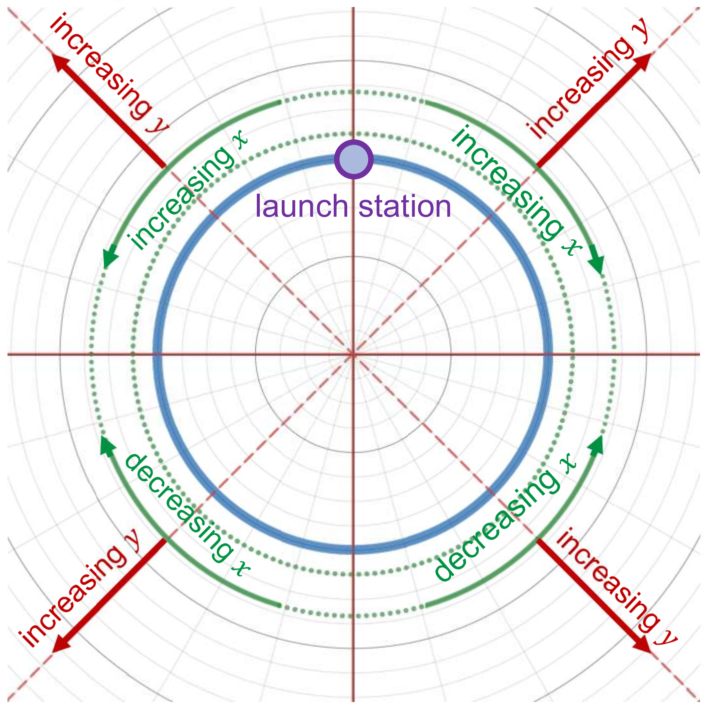

Figure 8: Illustration of axis definition (blue: Earth, purple: launch point, red: y-axis, green: x-axis) (Desmos, Inc., 2020)

BI-DIRECTIONAL VELOCITY AND ANGLE CALCULATIONS

Now that my directions are defined and the altitude and velocity curves have cartesian approximations, I will use differential calculus to discover more flight path properties. It was known that $v(t) = \frac{d}{dt}s(t)$ when v is velocity, s is displacement and t is time (Elert, 2020). Thus, I found vertical velocity against time as the derivative of vertical displacement (altitude) against time. However, I had an issue with non-differentiability from this function's discontinuity.

Figure 9 shows $s_y(t)$ after I zoomed in on the piecewise boundary at the commencement of flight stage 2 at $t = 196$.

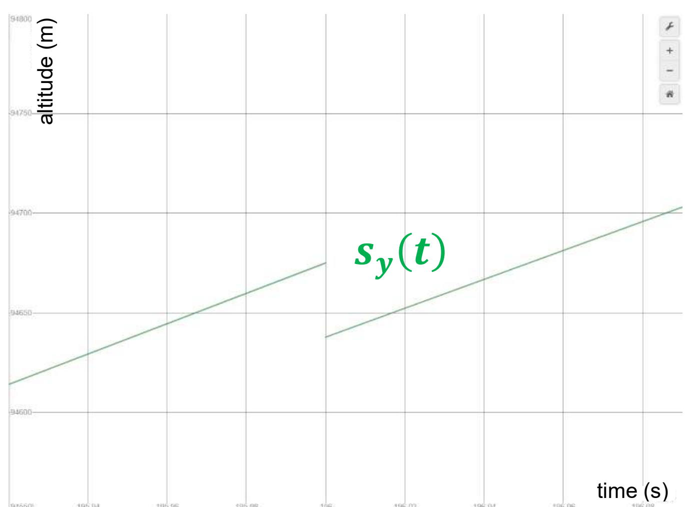

Figure 9: Close analysis of the 3rd to 4th data subset jump in altitude vs. time (Desmos, Inc., 2020)

For me to differentiate this function, the function must be differentiable. Continuity at a point is a necessary condition for differentiability at that point. For this to occur at point $x = a$ on function $f(x)$, the following condition must be satisfied (Fannon, Kadelburg, Woolley, & Ward, 2013):

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = f(a)$$

I see this to mean that at point $x = a$, there must not be a sudden change in the function's value without a continuous slope between. However, for the boundary of $s_y(t)$ seen in Figure 9, I found that $\lim_{t\to 196^-} s_y(t) \neq \lim_{t\to 196^+} s_y(t)$ since $\lim_{t\to 196^-} s_y(t) \approx 94674.228$ and $\lim_{t\to 196^+} s_y(t) \approx 94637.795$. This was the same for all 5 other piecewise boundaries in $s_y(t)$ and $v(t)$. This means that neither of these functions are continuous across $0 \leq t \leq 500$. Then, from the definition of differentiability, I unfortunately deduced that neither of them were differentiable either.

I saw this non-differentiability as a serious issue for my calculus-based analysis and it confused me for a considerable amount of time. However, I eventually bypassed this by redefining $s_y(t)$ and $v(t)$ in terms of their component functions below, where $f_n(x)$ is the n^{th} component of piecewise $f(x)$.

$$s_y(t) = \begin{cases} s_{y1}(t), & 0 \leq t < 153 \\ s_{y2}(t), & 153 \leq t < 186 \\ s_{y3}(t), & 186 \leq t < 196 \\ s_{y4}(t), & 196 \leq t \leq 500 \end{cases} \qquad v(t) = \begin{cases} v_1(t), & 0 \leq t < 153 \\ v_2(t), & 153 \leq t < 186 \\ v_3(t), & 186 \leq t < 196 \\ v_4(t), & 196 \leq t \leq 500 \end{cases}$$

From this, I defined $v_y(t) = \frac{d}{dt} s_y(t)$, I first differentiated its four component functions individually and then reformed the piecewise function under the same boundary domains as below:

$$v_y(t) = \begin{cases} v_{y1}(t) = s_{y1}'(t), & 0 \leq t < 153 \\ v_{y2}(t) = s_{y2}'(t), & 153 \leq t < 186 \\ v_{y3}(t) = s_{y3}'(t), & 186 \leq t < 196 \\ v_{y4}(t) = s_{y4}'(t), & 196 \leq t \leq 500 \end{cases}$$

As seen in Figure 10, as the rocket accelerates, vertical velocity increases. However, due to the turn in its path, the Falcon Heavy's vertical velocity returns to zero at its maximum altitude. I believe this is because of the new x directional velocity.

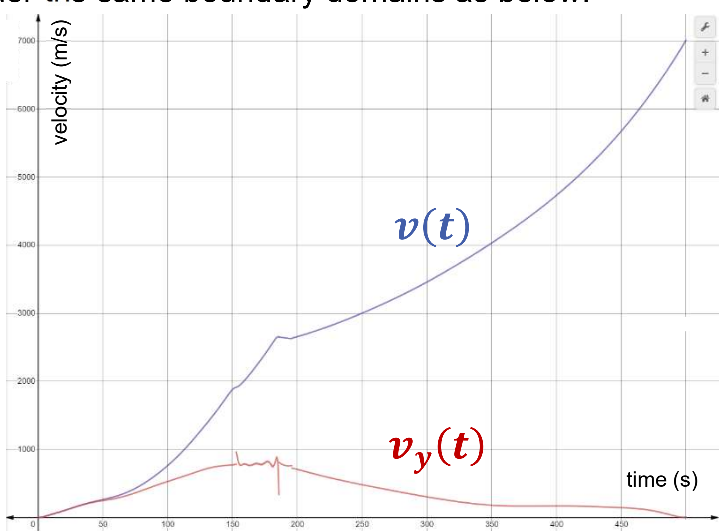

Figure 10: Comparison of velocity vs. time and vertical velocity vs. time graphs (Desmos, Inc., 2020)

I also immediately noticed prominent discontinuities in $v_y(t)$ at the points $t = 153, t = 186, t = 196$. However, I decided to pass these off as mathematical aberrations due to my method of bypassing non-differentiability. This is because they imply sudden changes in $s_y(t)$'s gradient; which did occur. This is a benefit of my cartesian visual model, being accessible and easily adjustable, while a point-by-point analysis, where one equation is applied to each data point, may not recognise or investigate this impossible velocity change.

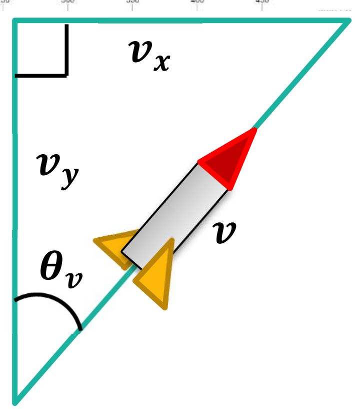

Figure 11: Vertical and horizontal velocity components

My next stage is computing velocity in the x direction with respect to time, which I define as $v_x(t)$. Based on Figure 11 for a v velocity rocket travelling at angle θ_v from the y direction, the Pythagorean theorem may be used to calculate $v_x(t)$ from $v_y(t)$ and $v(t)$ with the equation:

$$v_x(t) = \sqrt{\left(v(t)\right)^2 - \left(v_y(t)\right)^2}\ \{v(t) \geq v_y(t)\}$$

However, with Figure 12 showing the first 50 seconds of $v_x(t)$, I noticed that some intervals were undefined, just before the function increases significantly. This issue was confusing more me, so I revisited the launch.

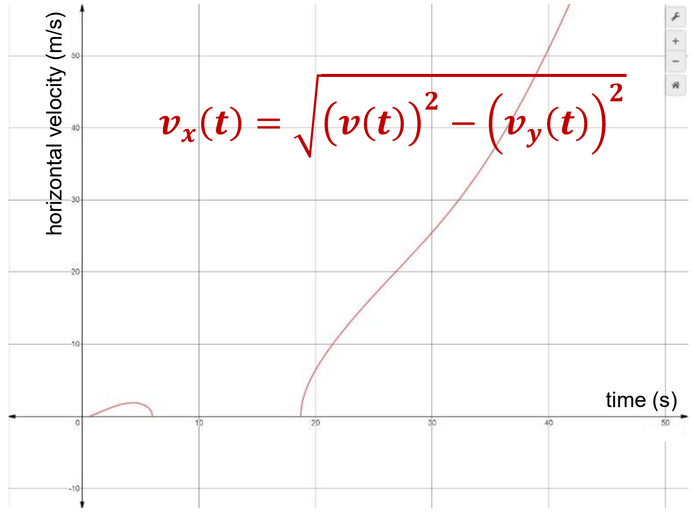

Figure 12: First 50 seconds of horizontal velocity vs. time without tool error correction (Desmos, Inc., 2020)

After watching the launch webcast again, I realised that in the interval before $t = 18.756$, the Falcon Heavy had not even begun its turn. As such, if any tool error in velocity or altitude measurements caused the measured $v_y(t)$ to exceed $v(t)$ in this time, then my use of Pythagoras' theorem would be undefined. As such, from intuition, to repair this issue, I redefined $v_x(t)$ to be equal to 0 before the rocket turned. The new graph is shown in Figure 13 and the piecewise equation appears below:

$$v_x(t) = \begin{cases} 0, & 0 \leq t < 18.756 \\ \sqrt{\left(v(t)\right)^2 - \left(v_y(t)\right)^2}, & 18756 \leq t \leq 500 \end{cases}$$

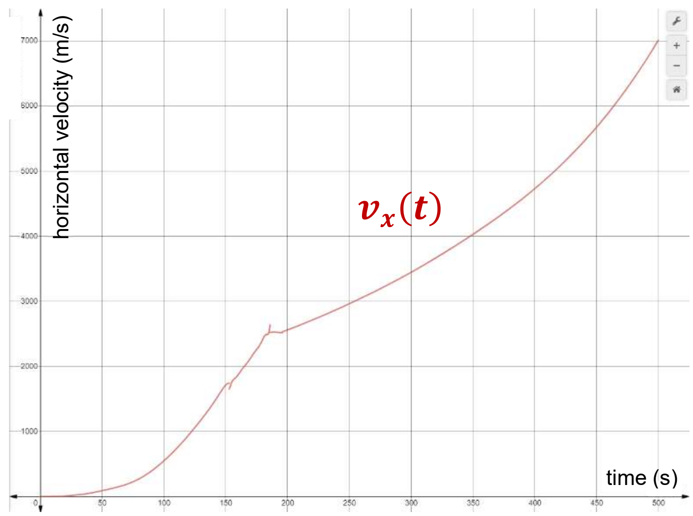

Figure 13: Horizontal velocity vs. time with tool error correction (Desmos, Inc., 2020)

Reflecting on Figure 11, I realised that a useful way to examine the instantaneous trajectory of the rocket would be to form a curve of θ_v against time. Using the trigonometric identity of tangent, in Figure 11 meaning that $\tan(\theta) = \frac{v_x}{v_y}$, I defined $\theta_v(t)$ such that $\theta_v(t) = \arctan\left(\frac{v_x(t)}{v_y(t)}\right)$ and this is plotted in Figure 14. I once again overlooked the remnants of non-differentiability manipulation at the piecewise domain boundaries. With this consideration, it is clear that the curve plateaus at $\theta_v(500) \approx 1.5701^c \approx 89.96°$. This confirms that the rocket initially flies vertically before slowly beginning to follow the x direction.

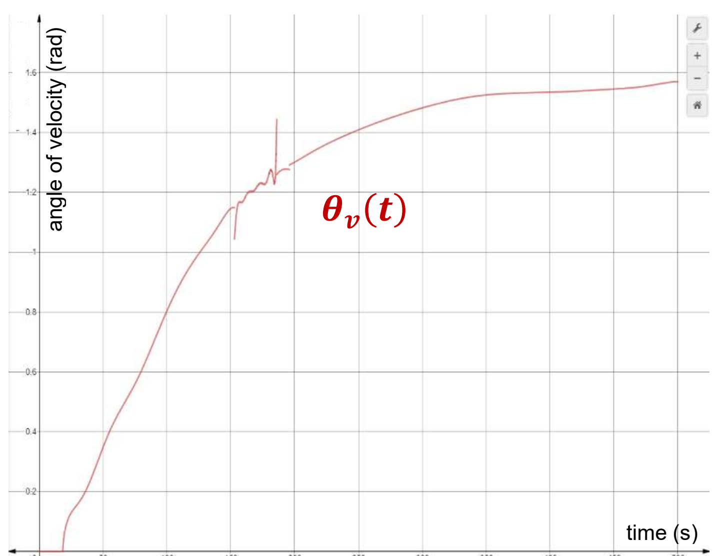

Figure 14: Velocity angle from y-direction vs. time (Desmos, Inc., 2020)

EARTH SURFACE MOTION PROJECTION

When SpaceX tracks the motion of their rockets to retrieve them for reusability, which they are leading the world in, flight profiles are used (SpaceX, 2020). Flight profiles are graphical plots of altitude against a quantity called downrange distance. In essence, downrange distance is the distance along the surface of the earth that the rocket has travelled from the launch site (National Aeronautics and Space Administration, 2020). I have developed my own method to mathematically derive downrange distance with respect to time for the Falcon Heavy Test Flight, using the information that I have.

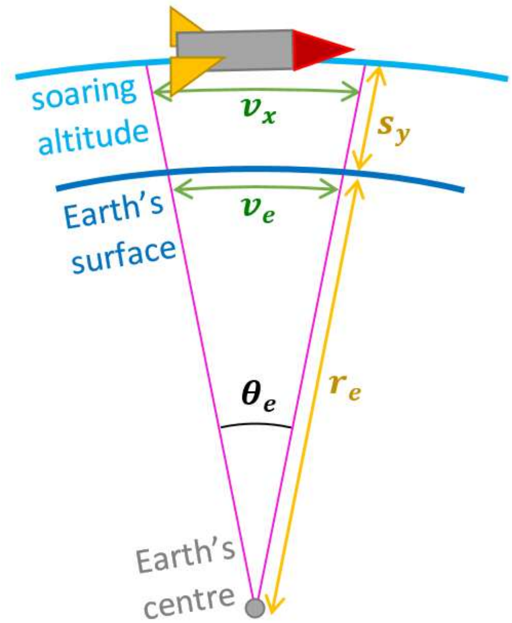

Figure 15: Earth surface velocity visualisation from arc length formula

In Figure 15, I show a rocket soaring at a distance equal to $r_e + s_y$ from Earth's centre. This is the radius of the Earth and the rocket's altitude. I also show v_e, the projection of distance v_x onto the surface of the Earth at radius r_e. Since v_x is the x direction distance the rocket travels in one second, v_e is effectively the velocity of the rocket on the Earth's surface. v_e is smaller than v_x when s_y is positive because across a constant Earth angle of θ_e, a concentric circular sector of larger radius will have a longer arc length.

In one second, v_x is small enough that I can approximate it as an arc length around the centre of the Earth at radius $r_e + s_y$ across angle θ_e (Tarquin Group, 2020). The arc length formula states that this arc length is $v_x = \theta_e(r_e + s_y)$ for θ_e in radians (Purplemath, 2020). In Figure 15, I demonstrated two arc lengths, including that of v_x. However, I also have that $v_e = \theta_e r_e$. By rearranging these and by the fact that the sweeping angle θ_e stays constant, I found the equality:

$$\theta_e = \frac{v_x}{r_e + s_y} = \frac{v_e}{r_e}, \therefore v_e = \frac{r_e}{r_e + s_y} v_x$$

From this ratio, I defined Earth surface velocity of the Falcon Heavy against time, $v_e(t)$, when $r_e = 6371000m$, as $v_e(t) = \frac{r_e}{r_e + s_y(t)} v_x(t)$. I used the mean radius of the Earth, calculated by volume. $v_e(t)$ is compared to $v_x(t)$ in Figure 16 (Williams, 2020).

Evidently, the two curves begin together, yet as time goes on, the horizontal velocity exceeds the equivalent velocity on the Earth's surface. This makes sense to me because with a higher altitude, the ratio coefficient used to find $v_e(t)$ decreases.

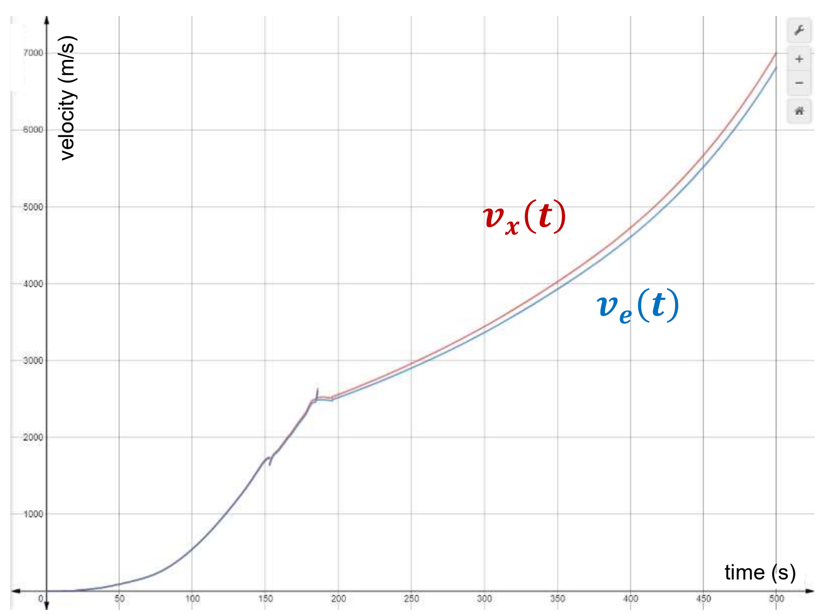

Figure 16: Comparison of horizontal and surface velocity vs. time (Desmos, Inc., 2020)

Since $s'(t) = v(t)$, then $\int v(t)dt = s(t) + c$ (Elert, 2020). Thus, for the Falcon Heavy's downrange distance $d_{down}(t)$ (its total distance travelled on the Earth's surface), I think $d_{down}(t) = \int v_e(t)dt$. I tried to define $d_{down}(t) = \int_{x=0}^{x=t} v_e(x)dx$ in Desmos, however, while it initially worked each time, Desmos could not process all of the calculations and the software crashed multiple times.

This was momentarily frustrating for me, until I realised I could use integral approximation. I visualised the definite integral $\int_{x=a}^{x=b} f(x)dx$ as the sum all infinitely thin width (dx) rectangles under the curve $f(x)$ between $x = a$ and $x = b$. Therefore, since $\int_{x=a}^{x=b} f(x)dx = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x$, I simply removed the limit statement and increased the rectangles' widths to a time interval of one second (a small interval for space travel). This produced that $d_{down}(t) = \sum_{x=0}^{x=t} v_e(x)$. I have put this function in Figure 17, but when looking closely, appears as 500 horizontal lines of length 1 second. So, I treated these lines as data points, fitting another polynomial regression to gain the smooth curve seen in Figure 18. Once again, I set the constant term, a_0, to 0, so to maintain that $d_{down}(t) \geq 0$.

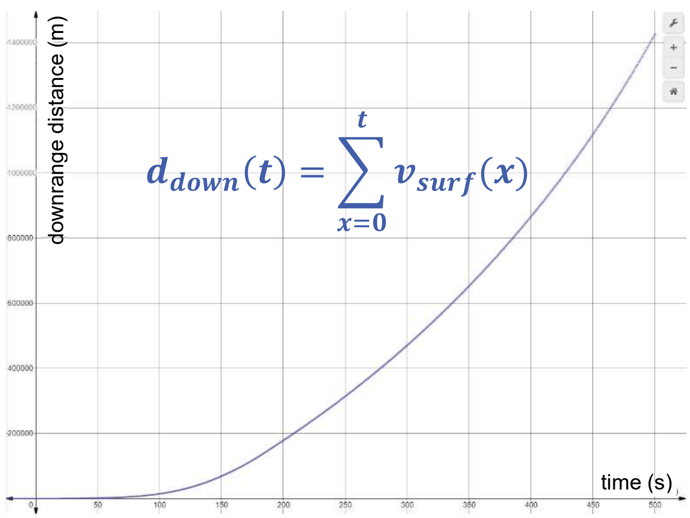

Figure 17: Downrange distance vs. time sum approximation (Desmos, Inc., 2020)

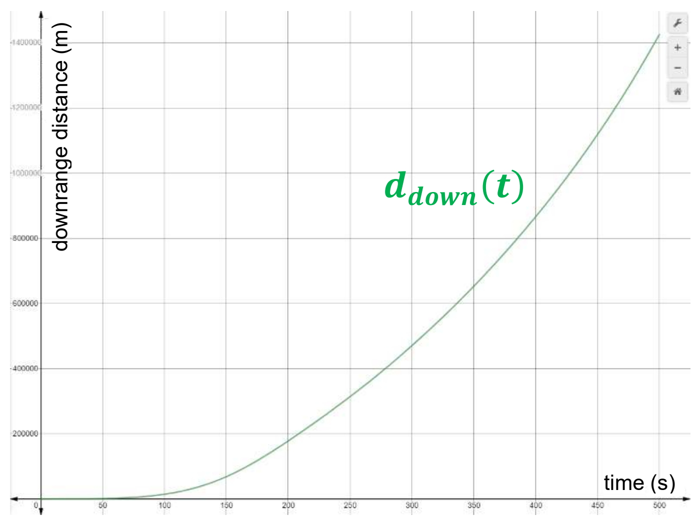

Figure 18: Downrange distance vs. time polynomial regression (Desmos, Inc., 2020)

PARAMETRIC FLIGHT PROFILE AND ARC LENGTH

The parametrisation of a curve is when the x axis variable and y axis variable are treated as two independent functions of one auxiliary parameter, usually t (School of Mathematics and Physics, The University of Queensland, 2020). A parametric curve on a cartesian plane is such that $(F(t), G(t))$ are the coordinates of each point. The individual functions $x = F(t)$ and $y = G(t)$ are said to parametrise the curve (School of Mathematics and Physics, The University of Queensland, 2020). Figure 19 illustrates this parametric definition in the x and y coordinate movements as t changes.

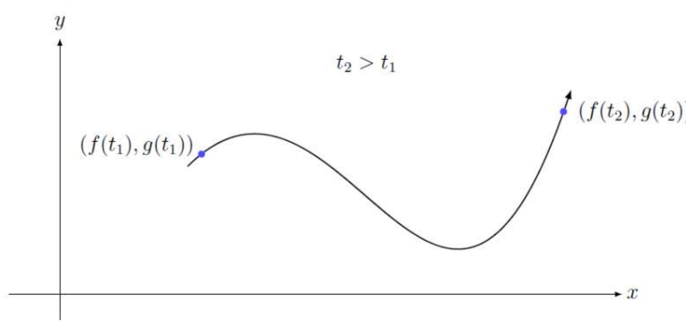

Figure 19: Basic illustration of curve parametrisation (School of Mathematics and Physics, The University of Queensland, 2020)

I find this theory extremely interesting as it seems like the classical Cartesian functions that we use, such as $y = G(x)$, are just a simplified version of a more general parametric case. This is the case of parametric equation $\begin{cases} x = F(t) \\ y = G(t) \end{cases}$ when $x = F(t) = t$, thus simplifying $y = G(t)$ to be $y = G(x)$.

From this definition of a parametric equation and the fact that a flight profile is the plot of altitude against downrange distance, I believe that for the Falcon Heavy Test Flight, I have found its flight profile to be the parametric equation:

$$\begin{cases} x = d_{down}(t) \\ y = s_y(t) \end{cases} \{0 \le t \le 500\}$$

I show this curve in Figure 20. For me, this is an incredibly impressive result, because this parametric flight profile shows what the path of the Falcon Heavy would have looked like, had the Earth been flat and ran across the x axis. I find that really cool!

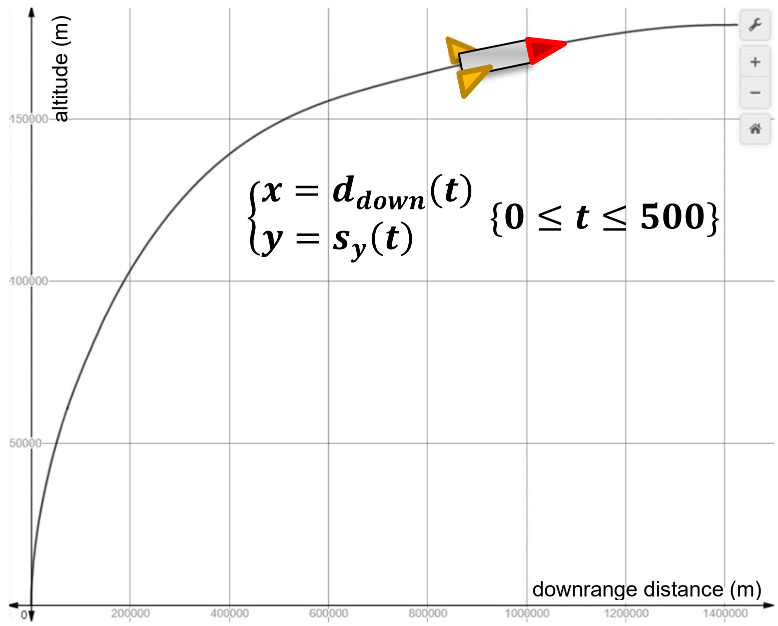

Figure 20: Parametric flight profile of Falcon Heavy ascension (Desmos, Inc., 2020)

I now want to find the total distance travelled by the Falcon Heavy in ascension, which I think should be the total length of the parametric curve in Figure 20. Similar to my visualisation of a definite integral, I imagined the length L between $t = a$ and $t = b$ in a parametric curve as the sum of the lengths of all infinitely short lines in this interval. From here, I applied Pythagoras' theorem to find these lengths by an infinitely small change in the x coordinate and y coordinate.

Thus, for parametric equation: $\begin{cases} x = F(t) \\ y = G(t) \end{cases}$, $L = \lim_{\delta t \to 0} \sum_{t=a}^{t=b} \sqrt{\left(\mathrm{F}(t + \delta t) - F(t)\right)^2 + \left(\mathrm{G}(t + \delta t) - G(t)\right)^2}$

From here, I remembered that for a line, the gradient, $m = \frac{\Delta y}{\Delta x}$, and hence, $\Delta y = m\Delta x$. The derivative of a curve is a continuous representation of its gradient, so I thought $F(t + \delta t) - F(t) = F'(t) \cdot \delta t$ when δt approaches 0. After later learning Euler's method for numerical integration, I was astonished that my logic was simply a variation of Euler's method when the step value approaches 0 (Blythe, et al., 2005). As I continued to develop my curve length equation below by this substitution, this discovery gave me great confidence in my procedure, also prompting further work.

For $\begin{cases} x = F(t) \\ y = G(t) \end{cases}$, $L = \lim_{\delta t \to 0} \sum_{t=a}^{t=b} \sqrt{(\mathrm{F}'(\mathrm{t})\delta t)^2 + (G'(t)\delta t)^2} = \lim_{\delta t \to 0} \sum_{t=a}^{t=b} \sqrt{\left(F'(t)\right)^2 + \left(G'(t)\right)^2}\,\delta t$

Finally, I used the same logic as for integral approximation once again, where I converted the above sum into an integral. I was pleased when this gave me the final equation that the length of the parametric curve $\begin{cases} x = F(t) \\ y = G(t) \end{cases}$ between $t = a$ and $t = b$ was $L = \int_{t=a}^{t=b} \sqrt{\left(F'(t)\right)^2 + \left(G'(t)\right)^2} dt$.

I was even more pleased after research when I found that my own derivation was not only correct, but had its own theory call the parametric arc length equation. As a result, for the Falcon Heavy's flight profile $\begin{cases} x = d_{down}(t) \\ y = s_y(t) \end{cases}$ in $0 \le t \le 500$, I performed my own calculation. This enabled me to find that $L = \int_{t=0}^{t=500} \sqrt{\left(\frac{d}{dt} d_{down}(t)\right)^2 + \left(\frac{d}{dt} s_y(t)\right)^2}\, dt \approx 1461715m$; the total ascension distance travelled by the Falcon Heavy (School of Mathematics and Physics, The University of Queensland, 2020).

POLAR CONVERSION FOR FLIGHT PROFILE

However, I was still unsatisfied with this result, as it left the assumption that the Earth's surface was flat and I wanted a way to show the Falcon Heavy's real trajectory. So, I imagined a system where the Earth's centre was the origin of a cartesian plane, so that the Falcon Heavy's position could be shown as coordinates in relation to Earth.

With my data, I immediately jumped to representing the Falcon Heavy's position as a complex number on an Argand plane. This is because, when put in polar form, the complex number z appears as $z = |z|cis(\theta) = rcis(\theta)$ where r is z's displacement from the origin and θ is the sweeping angle of z from the real plane (Urban, et al., 2008). To use this coordinate system based on magnitude and argument for the Falcon Heavy, I defined my unmoving 'cosmic' x and y axes to be the real and imaginary planes, respectively. From this, I had the magnitude of each polar rocket position from Earth's centre as $r = r_e + s_y$. From the arc length of a sector equation, I also found the angle around the Earth that the rocket had travelled after launch to be $\frac{d_{down}}{r_e}$. Thus, at any moment in time, the Falcon Heavy's polar position was $z(t) = |r_e + s_y(t)|cis\left(\frac{\pi}{2} - \frac{d_{down}(t)}{r_e}\right)$.

Since the rocket did not travel at least a quarter way around the Earth, I used $\frac{\pi}{2} - \frac{d_{down}(t)}{r_e}$ as the argument instead of $\frac{d_{down}(t)}{r_e}$. This was purely graphical, so that I could reflect the vertical launch as positive motion in the imaginary axis.

Figure 21 shows how polar complex numbers are given cartesian coordinates. For complex number $z = rcis(\theta)$, the real plane coordinate is $rcos(\theta)$ and the imaginary plane coordinate is $rsin(\theta)$ (Urban, et al., 2008).

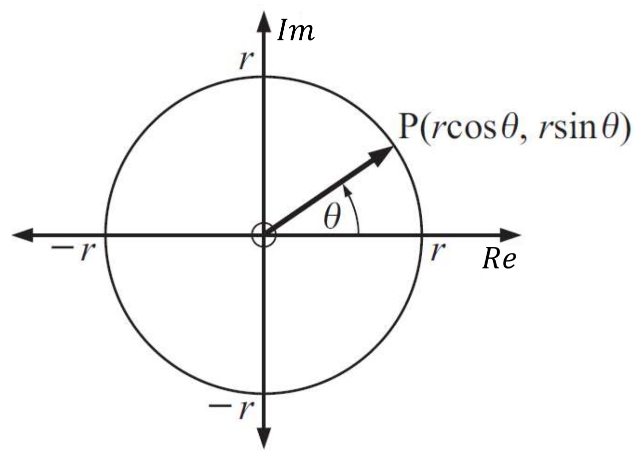

Figure 21: Real and imaginary coordinates of a polar complex number (Urban, et al., 2008)

From this, I made the two real and imaginary coordinate functions for the Falcon Heavy after launch when $r_e = 6371000m$ as the Earth's previously stated mean radius by volume (Williams, 2020):

$Re(t) = \left(s_y(t) + 6371000\right)\cos\left(\frac{\pi}{2} - \frac{d_{down}(t)}{6371000}\right)$ and $Im(t) = \left(s_y(t) + 6371000\right)\sin\left(\frac{\pi}{2} - \frac{d_{down}(t)}{6371000}\right)$.

Since I had two independently operating axes based on one parameter, t, I built the parametric equation $\begin{cases} x = Re(t) \\ y = Im(t) \end{cases} \{0 \leq t \leq 500\}$ from this real plan x axis and imaginary plane y axis .

This parametric curve is Figure 22 with a circle of radius r_e to represent the Earth. I was satisfied with this result, mostly based on how visually pleasing it looks. I also took another arc length of my new curve:

$$L = \int_{t=0}^{t=500} \sqrt{\left(Re'(t)\right)^2 + \left(Im'(t)\right)^2}\,dt$$

$$\therefore L \approx 1494191m$$

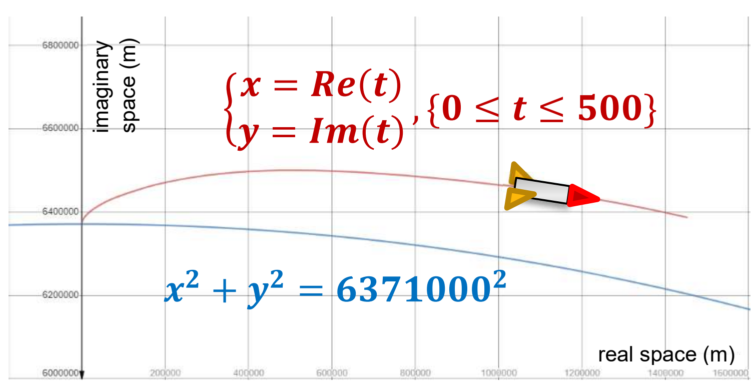

Figure 22: Falcon Heavy Test Flight (blue: earth, red: flight path) (Desmos, Inc., 2020)

METHOD ACCURACY EVALUATION

While the result above was visually pleasing, it surprised me that the arc length of the polar diagram was almost 3000m larger than that of the flight profile. This inherently indicates a degree of error, meaning it is important to assess the accuracy of my results. Since SpaceX does not release their own analysis of their telemetry data, I can only assess my result's accuracy against the other analyses from the space exploration community. I was able to find various examples of community analysis, and while these may be prone to their own errors, they are the only gauge of accuracy and source of comparison that I have. However, the Falcon Heavy Test Flight is well-studied and mathematically investigated, so I believe that I can rely on the community's calculations to assess the effectiveness of my method with reasonable confidence.

In all of the external analyses I found, downrange distance was found and this was plotted against altitude to give a flight profile, meaning this remained the most important conclusion for me to compare my results from Figure 20 with. I also chose a form of analysis that does not rely on time, since other analyses may have recorded data from different times. I performed a simple comparison between my value for downrange distance at the maximum altitude of $s_y(500) \approx 178967m$, being $d_{down}(500) \approx 1424725m$ and the same for other analyses.

The three downrange distance values I found online, correlating to an altitude of 178967m, were 1454871m, 1445554m and 1438756m. The latter two values were not in fact calculations from the telemetry, but predictions prior to the flight based on the information that SpaceX had published. Since these predictions are likely to be less accurate than the data analysis, when taking an average to be used as a literature value, the first value was weighted as 50% of the mean, while the other two were weighted at 25% each. This allowed the measured performance of the rocket to be demonstrated by this figure. As such, the community-predicted downrange distance at my maximum altitude was $\overline{d_{down}} = \frac{2\times1454871+1445554+1438756}{4} \approx 1448513m$. Taking this calculation as a literature value, my calculation is approximately 24km lower, a percentage difference of -1.64%. In reality, for space travel, this is very accurate and I was ecstatic to find that my method was successful. However, I believe that it is not a coincidence that all of the literature community values were higher than mine by at least 10km.

This may have been due to one of my early assumptions. In neglecting a third dimension of movement for the rocket, I assumed that all lateral motion would be negligible. However, in reality, it is almost inevitable that the Falcon Heavy would have moved in all three directions. If this had occurred, my analysis would not account for this lateral movement, causing my total downrange distance to be slightly smaller than the true value.

METHOD ACCESSIBILITY AND APPLICATIONS

Although this planar inaccuracy serves as a flaw in my method, the cartesian procedure that I used has benefits in its accessibility and applications which validify it for further use. Firstly, by conducting my analysis in a cartesian form with function manipulation, at each stage of analysis, changes in the data can be visibly seen. This allows effective human judgment over issues such as fluctuations due to bypassed undifferentiability, while an objective data analysis may propagate these issues, expanding error. This use of graphs in my analysis is then proven beneficial by its effectiveness in the visual tracking of a moving object that can be easily interpreted by anyone.

Additional, in the data science study of computational complexity, the "Big O Notation" of a mathematical algorithm is the relative time needed for an output based on input size (Batista, 2018). I believe that, since I processed all points from the initial data set in the same manner, it is likely that my algorithm would have an $O(n)$ notation. This means that if the number of input elements were increased by a factor of n, then the time needed to generate an output would be scaled by n as well, being judged as fairly efficient by the legend in Figure 23. This is advantageous as it is extremely accessible to all investigators in the field.

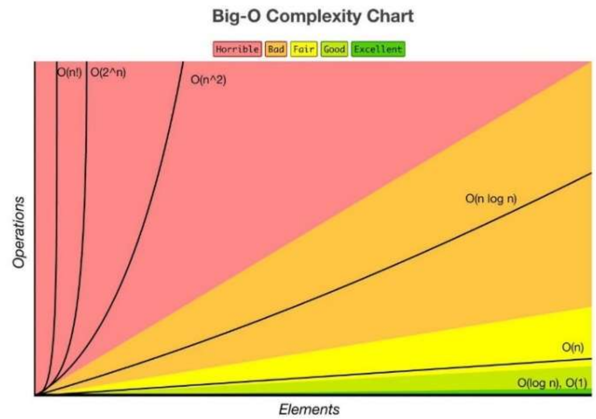

Figure 23: Big O Notation explanation visual (Batista, 2018)

Finally, although the limits of Desmos as an application are the manipulation of two-dimensional data, my mindset when approaching the telemetry data for the launch could easily be transferred into three-dimensional analysis using the study of spherical coordinates. As an extension of my use of polar coordinates, relying on a magnitude and sweeping angle, spherical coordinates, as seen in Figure 24, rely on a magnitude, directional angle between two dimensions and a third angle of elevation into the third dimension to define the position of an object in three-dimensional space (called the radial distance, r, polar angle, θ, and azimuthal angle, ϕ, respectively) (Wolfram Research, Inc., 2020). Hence, my analysis finds more complex applications in the trajectory analysis of three-dimensional space travel.

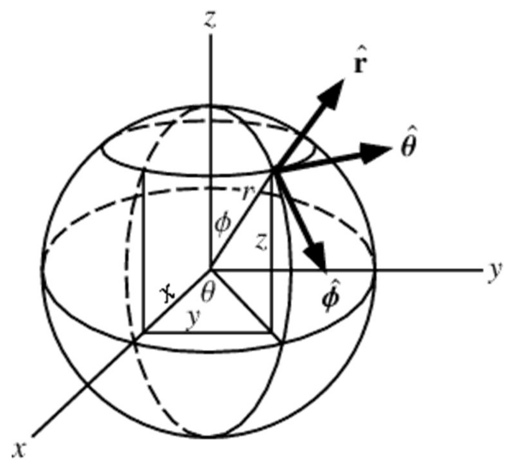

Figure 24: Spherical coordinates visual (Wolfram Research, Inc., 2020)

CONCLUDING STATEMENTS

In this exploration, I took a personal interest of mine, being the trajectory tracking of rockets, in the context of the Falcon Heavy Test Flight from 2018, to generate a mathematical analysis of telemetry data released by SpaceX. I was able to develop a method by which I could calculate various kinematic quantities as functions of time after the launch of the Falcon Heavy, before then using parametric equations to show the rocket's continuously changing position after launch. My method was not only successful, able to calculate some of the most valuable pieces of information for the future of rocket reusability to a percentage proximity of 1.64%, but also is accessible to all mathematical enquirers through its reasonably efficient Big O Notation. I was also able to reflect on my algorithm's potential compatibility with spherical coordinates, being able to track the trajectory of much more complicated rocket launches which have motion in three-dimensional space through the use of parametric equations defined by x, y and z axes. As a result, with some adjustments for the third positional axis, my method for trajectory analysis, based on cartesian regression analysis and supplemented by my knowledge of SpaceX and the Falcon Heavy, could prove effective for the investigation into future space exploration missions.

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