Computing the current response of linear passive circuits to arbitrary voltage waveforms using Fourier analysis

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March 12, 2025

1 Abstract

This experiment pursued a systematic approach to quantitatively understand the behaviour of simple linear circuits, given the possibility of an arbitrary input voltage. A voltage waveform is approximated by piecewise list of linear functions, for which an analytic solution can be found for its complex Fourier series coefficients. A program could then decompose the voltage waveform into a discrete set of AC components, without the need for computing integrals. By exploiting the superposition properties of linear circuits, the effect on current of each voltage component was analysed independently. The scope of this experiment was LCR circuits, but the results indicate applicability for any possible arrangement.

Despite the predefined scope of Fourier series to periodic functions, non-repeating signals of finite time length could have their transient effects computed. Furthermore, the method's usefulness was extended beyond calculating currents from voltages, allowing the theoretical calculation of any quantities in circuits which could be found with complex methods, given sinusoidal time dependence. This proved it to be more versatile, efficient and systematic than the standard methods of circuit analysis combined, being ordinary differential equations and complex impedance analysis.

2 Introduction

2.1 Differential equations method

Classically, circuits made of passive devices are solved with differential equations. Figure 1 demonstrates an LCR circuit, which consists of an inductor, L, capacitor, C, and resistor, R, in series, stimulated by a voltage waveform $V(t)$, with resulting current waveform $I(t)$.

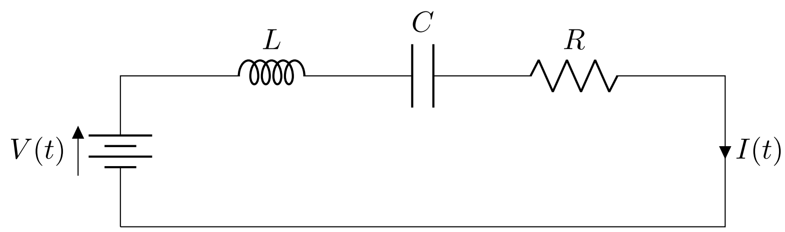

Figure 1: LCR circuit with arbitrary voltage waveform

Using Kirchoff's voltage law and formulae for the potential difference across these passive devices, we find that the charge on the capacitor, $Q(t)$, follows:

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}(t) + R\frac{\mathrm{d}Q}{\mathrm{d}t}(t) + \frac{1}{C}Q(t) = V(t) \quad \text{where} \quad I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t}(t).$$

With an exponential function ansatz, and defining $\gamma \equiv R/2L$ and $\omega_0 \equiv 1/\sqrt{LC}$, we find three complementary functions for this second-order inhomogeneous ordinary differential equation (Smith, 2022):

- Overdamped ($\gamma > \omega_0$): $Q_{CF}(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0{}^2}t} + Be^{-\sqrt{\gamma^2-\omega_0{}^2}t}\right)$
- Underdamped ($\gamma < \omega_0$): $Q_{CF}(t) = e^{-\gamma t}\left(A\cos\sqrt{\gamma^2-\omega_0{}^2}t + B\sin\sqrt{\gamma^2-\omega_0{}^2}t\right)$
- Critically damped ($\gamma = \omega_0$): $Q_{CF}(t) = e^{-\gamma t}\left(At + B\right) \quad \text{for} \quad A, B \in \mathbb{R}$.

All symbolise an exponentially decaying transient response. The particular integral of the inhomogeneous term provides additional transient effects, or a steady-state solution in the case it is periodic. A simple example is a stepped voltage of the form $V(t) = V_0\,\theta(t)$, where θ is the Heaviside theta function. To solve this, we trial the constant solution $Q_{PI}(t) = Q_0$, finding that $Q_0 = CV_0$. Applying the initial conditions that $Q(0) = 0$ (cannot instantaneously charge capacitor) and $I(0) = 0$ (cannot instantaneously introduce inductor current) gives the values of A and B, depending on the damping case. For example, in the underdamped case, we differentiate and reach the final current:

$$I(t) = \frac{V_0}{L\sqrt{{\omega_0}^2 - \gamma^2}} e^{-\gamma t} \sin\left(\sqrt{{\omega_0}^2 - \gamma^2}t\right)\theta(t).$$

Another basic example is for a harmonic voltage, such as $V(t) = V_0 \cos \omega t$. Inserting the oscillatory anstaz $Q_{PI}(t) = C \cos \omega t + D \sin \omega t$ into the differential equation, we find values of C and D:

$$C = \frac{V_0}{L}\frac{{\omega_0}^2 - \omega^2}{\left({\omega_0}^2 - \omega^2\right)^2 + (2\omega\gamma)^2} \quad \text{and} \quad D = \frac{V_0}{L}\frac{2\omega\gamma}{\left({\omega_0}^2 - \omega^2\right)^2 + (2\omega\gamma)^2}.$$

We choose to neglect the transient response, meaning $Q(t) = Q_{PI}(t) + \cancel{Q_{CF}(t)}$. Further, we wish to write the solution in the form $Q(t) = A \sin(\omega t + \phi)$ for some A and ϕ. By using the relations, $D/C = -\tan\phi$ and $C^2 + D^2 = A^2$, and then differentiating, we find the steady-state current to be:

$$I(t) = \frac{[-]\,\omega V_0}{L\sqrt{\left({\omega_0}^2 - \omega^2\right)^2 + (2\omega\gamma)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{2\omega\gamma}{{\omega_0}^2 - \omega^2}\right) + \frac{\pi}{2}\right)$$

with $[-]$ if $\omega > \omega_0$ and $I(t) = \frac{V_0}{R}\cos\omega t$ if $\omega = \omega_0$.

2.2 Complex AC impedance method

One can observe, in the harmonic example, that both $V(t)$ and $I(t)$ can be expressed as the real component of a complex phasor, $\nu e^{i\omega t}$, for some complex ν. If we denote these phasors by $V(t) = \Re\left(\tilde{V}(t)\right)$ and $I(t) = \Re\left(\tilde{I}(t)\right)$, we define the complex impedance, $Z(\omega)$, as:

$$Z(\omega) \equiv \frac{\tilde{V}(t)}{\tilde{I}(t)} = \frac{-iL}{\omega}\left(\left({\omega_0}^2 - \omega^2\right) + 2i\omega\gamma\right).$$

This is known as AC Ohm's law, relating the amplitude and phase of sinusoidal currents and voltages for a specified complex impedance. The circuit's impedance is calculated using the same rules for adding resistances in series and parallel, given that resistors, capacitors and inductors have impedances $Z_R = R$, $Z_C = 1/i\omega C$ and $Z_L = i\omega L$, respectively (Smith, 2022). Thus, for an LCR circuit, $Z = i\omega L + \frac{1}{i\omega C} + R$ which becomes the number found above.

This method is evidently more efficient for computing the steady-state current of this circuit from a sinusoidal voltage. However, it works for more circuits than just an LCR. Using impedance addition rules, the impedance of any arbitrary formation of resistors, capacitors and inductors can be calculated.

2.3 Limitations of a priori methods

Each of the above methods have strengths and limitations. The strength of the differential equation approach is that it is straight-forward to incorporate the voltage waveform into the differential equation. Furthermore, it ensures that transient responses are captured. On the other hand, acquiring the differential equation to begin with is difficult for more complicated arrangements of the passive devices, when the currents splits across parallel segments. This issue is solved by complex AC circuit analysis, which allows simple calculation of a complex impedance. However, this method is completely incompatible with voltage waveforms that are not explicitly sinusoidal, let alone periodic.

3 Methodology

3.1 Fourier circuit analysis method

The desire to harness the powers of both approaches justifies Fourier circuit analysis. By Fourier decomposing periodic voltage waveforms into sinusoidal terms, we can exploit the linearity of passive devices and theoretically attain its current response for any arbitrary voltage waveform. We begin by inspecting Figure 2, which is a circuit of complex impedance $Z(\omega)$, stimulated by voltage $V(t)$, resulting in current $I(t)$.

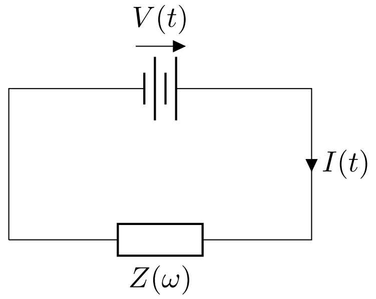

Figure 2: Complex impedance circuit with arbitrary voltage waveform

We begin by defining the complex voltage as $\tilde{V}(t) = V(t) + 0i$, meaning $V(t) = \Re\Big(\tilde{V}(t)\Big)$. This may seem pointless, but is important as the complex current $\tilde{I}(t)$ will differ from $I(t)$.

Given that $V(t)$ is periodic with period T, the next step is to expand $\tilde{V}(t)$ using the complex Fourier series. Details of this process are included in Appendix A, leading to the following result.

$$\tilde{V}(t) = \sum_{n\in\mathbb{Z}} \nu_n e^{i\omega_n t} \quad \text{where} \quad \nu_n = \frac{1}{T}\int_0^T e^{-i\omega_n t}\,\tilde{V}(t)\,\mathrm{d}t \quad \text{and} \quad \omega_n = \frac{2\pi n}{T}.$$

We define $\tilde{V}_n(t) \equiv \nu_n e^{i\frac{2\pi n}{T}t}$, so $\tilde{V}(t) = \sum_{n\in\mathbb{Z}} \tilde{V}_n(t)$. This enables us to redraw Figure 2 as in Figure 3, splitting the original voltage waveform into a set of independent AC sources.

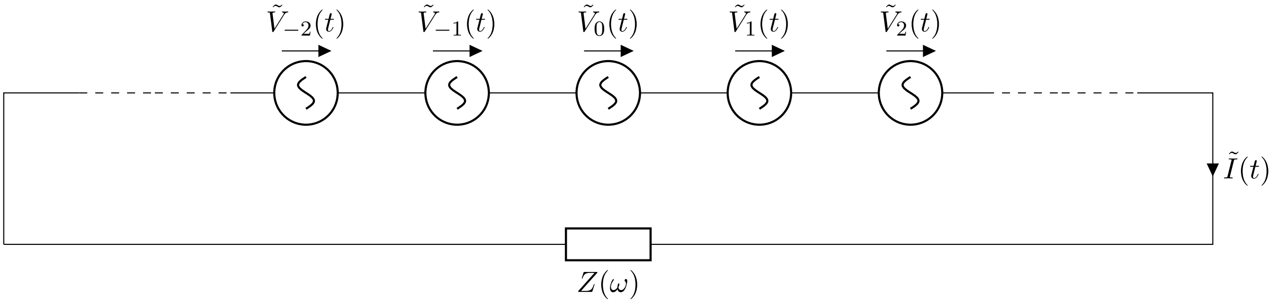

Figure 3: Complex impedance circuit with Fourier decomposed voltage source

It is here that we exploit the fact that circuits made solely of these passive devices are linear. In being linear, the principle of superposition applies to voltage sources and corresponding currents. This means that if a circuit is stimulated by a voltage $V(t) = V_A(t) + V_B(t)$, the resulting current will be equal to $I(t) = I_A(t) + I_B(t)$ where $I_A(t)$ is the current due to source A alone, and the same for B. It is for this reason that we can calculate the current, $\tilde{I}_n(t)$, due to each of the $\tilde{V}_n(t)$, and the sum of these is the final $\tilde{I}(t)$. Using the AC Ohm's law:

$$I(t) = \Re\Big(\tilde{I}(t)\Big) = \Re\left(\sum_{n\in\mathbb{Z}} \tilde{I}_n(t)\right) = \Re\left(\sum_{n\in\mathbb{Z}} \frac{\tilde{V}_n(t)}{Z(\omega_n)}\right).$$

3.2 Approximating the series

The method of computation, in theory, is to: determine $Z(\omega)$ for the circuit; calculate the Fourier coefficients for the input voltage, ν_n; tabulate them; apply the impedance $Z(\omega_n)$ to each; and recombine them to find the current. However, a computer cannot compute the infinite number of coefficients, so an approximation is made in choosing the number of coefficients to include. Furthermore, the calculation of many different ν_n involves many integrals, which entails long runtimes for a computer, whether done anaytically or numerically. We employ a further simplification to allow the computer to approximate each coefficient much more rapidly.

This method is explained in detail within Appendix B, but involves modelling the voltage waveform with a piecewise-linear function, defined by a series of straight lines that connect equally-spaced samples of the voltage. The complex Fourier series coefficients for this sort of piecewise linear function were calculated analytically, leading to the result that, for an input waveform $\tilde{V}(t)$ with period T, the Fourier decomposition up to Nth order, when the function is approximated with a list of P samples, is given by:

$$\tilde{V}_{P,N}(t) = \frac{1}{P}\sum_{k=0}^{P-1} z_k + \frac{1}{T}\sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{{\omega_n}^2}\left(\sum_{k=0}^{P-1} e^{-i\omega_n \frac{k}{P}T}(m_{k-1}-m_k)\right)e^{i\omega_n t}$$

where $z_k = \tilde{V}\left(\frac{k}{P}T\right)$ and $m_k = \frac{P}{T}(z_{k+1}-z_k)$.

Thus, with an impedance of $Z(\omega)$:

$$I_{P,N}(t) = \Re\left(\frac{1}{PZ(0)}\sum_{k=0}^{P-1} z_k + \frac{1}{T}\sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{{\omega_n}^2 Z(\omega_n)}\left(\sum_{k=0}^{P-1} e^{-i\omega_n \frac{k}{P}T}(m_{k-1}-m_k)\right)e^{i\omega_n t}\right).$$

It is noted that $\lim_{P,N\to\infty} I_{P,N}(t) = I(t)$.

3.3 The capacitor catastrophe

From the form of the current, it is evident that exceptions will be encountered if the impedance is 0. Furthermore, an impedance tending to infinity, due to a divide by 0 in $Z(\omega)$, cannot be computed. Recall that for an LCR circuit, the impedance is $Z(\omega) = R + i\left(\omega L - 1/\omega C\right)$. Ideally, we would be able to set $R = 0$, $L = 0$ or $C = \infty$ to remove the components and test the response of just an LR, LC or RC circuit.

Without a capacitor, setting $R = 0$ will always create issues for the DC component as $Z(\omega_0) = R$, reflecting that no real circuit has zero resistance. When this case is desired, the resistance was set to some sufficiently small number, so to have a comparably negligible impact.

The larger issue comes from the capacitor when there is a non-zero DC term. Problematically, despite many voltage waveforms having no DC term as the voltage analytically integrates to 0, almost all the time, the piecewise approximation of the function will integrate to a small, non-zero value, so long as P is finite. This means that when a finite capacitor exists, $Z(0)$ must be assumed to always tend to negative infinity, which a computer cannot handle.

As a result, an exception had to be made for the LCR circuit, with a manual, user-controlled, switch to indicate the presence of a finite capacitor. When enabled, the DC current term is forcefully set to vanish. For more complicated circuits, this sort of exception would need to be made whenever the impedance could be evaluated to infinity, which is problematic and makes automated calculation of the impedance a very complicated process.

It is for this reason that the scope of the experiment was refined to solely include the LCR circuit, however the impedance for more sophisticated circuits could be calculated by hand and inserted. Ultimately, the usefulness of this program is limited by the user's familiarity of AC circuit theory.

4 Results

We examine the operation of the model in comparison with the predictions made above, which were confirmed experimentally in the EL01 and EL02 practicals. For all of the tests, values of $N = P = 2500$ were chosen and the program halted within a few seconds. Throughout all tests, SI units were used.

4.1 Harmonic voltage

We begin with an intermediate set of LCR values, including $L = 0.005[\mathrm{H}]$, $C = 2[\mathrm{F}]$ and $R = 0.3[\Omega]$. Figure 4 illustrates one period of the result for a voltage waveform $V(t[\mathrm{s}]) = \sin t\,[\mathrm{V}]$, so $T = 2\pi[\mathrm{s}]$.

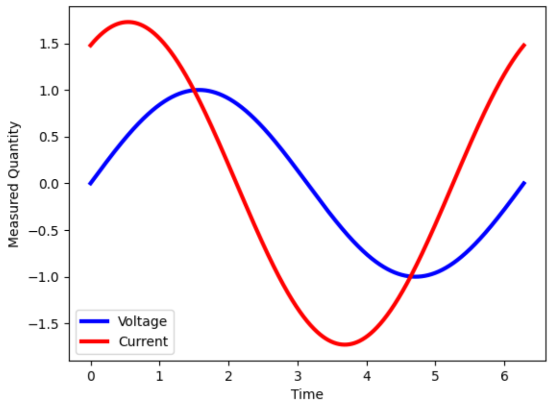

Figure 4: LCR response to $V(t[\mathrm{s}]) = \sin t\,[\mathrm{V}]$ with $L = 0.005[\mathrm{H}]$, $C = 2[\mathrm{F}]$ and $R = 0.3[\Omega]$

The current curve appears sinusoidal and sees a first peak of roughly 1.73[A] at 0.54[s]. The afforementioned formula for LCR impedance evaluates to roughly $0.5788e^{-1.0259i}$, predicting a first current peak of roughly 1.7277[A] at 0.5449[s]. These figures agree at the provided level of precision.

To test the model for the circuit's inductance limit, the same signal was used, but with a negligible resistor and no capacitor. Values of $R = 0.00001[\Omega]$ and $L = 0.5[\mathrm{H}]$ were chosen. The impedance formula tends to a value of $0.5i$ which indicates a quarter phase lag of current behind voltage and a doubling in amplitude. Figure 5 shows the results of the model.

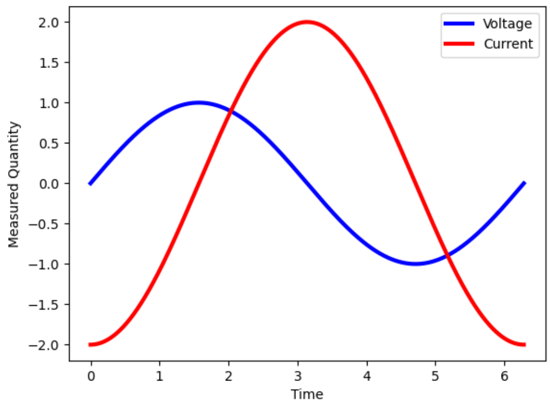

Figure 5: LR response to $V(t[\mathrm{s}]) = \sin t\,[\mathrm{V}]$ with $L = 0.5[\mathrm{H}]$ and $R = 0.00001[\Omega]$

With a clear lag of quarter phase and doubled amplitude, the model has successfully found the expected characteristics of this LR circuit and continues to perform for harmonic voltage sources.

4.2 Stepping voltage for underdamped LCR circuit

It appears as though, from the model's construction, that only periodic voltages are suitable. However, we attempt to analyse transient effects of non-repeating signals of finite length by fabricating a periodic signal which begins with the real signal, and then freezes sufficiently long before repeating to display all transient effects.

This approach was taken to test the stepping voltage. An input voltage was chosen of the form $V(t[\text{s}]) = \theta(-t-50)\,[\text{V}]$ with the repeating period set to between $t \in [0[\text{s}], 100[\text{s}])$. We also choose an underdamped circuit ($\gamma < \omega_0$), as it would be the easiest to compare to theory. To ensure this, the values chosen were $L = 1[\text{H}]$, $C = 0.5[\text{F}]$ and $R = 0.2[\Omega]$. Figure 6 demonstrates the output of the program.

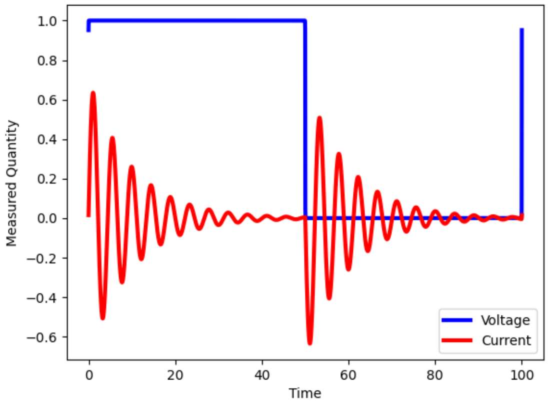

Figure 6: LCR response to $V(t[\text{s}]) = \theta(-t-50)\,[\text{V}]$ with $L = 1[\text{H}]$, $C = 0.5[\text{F}]$ and $R = 0.2[\Omega]$

As expected, Figure 6 exhibits characteristics of a sinusoidal function with exponentially decaying amplitude. From the graph, the space between 18 consecutive axes intercepts was roughly 40.04[s], implying an oscillating period of 4.45[s]. This period should be $2\pi({\omega_0}^2 - \gamma^2)^{-1/2}$ which evaluates to roughly 4.454[s].

Furthermore, based on the form of the solution, we expect the ratio of each peak's amplitude to the previous one to be $e^{-2\pi\gamma\left({\omega_0}^2-\gamma^2\right)^{-1/2}}$, which evaluates to roughly 0.6406. From the model, the ratio between one peak and the peak 4 prior was roughly 0.16, which implies a peak-to-peak ratio of 0.64. These figures indicate the model was successful reproducing predicted results.

This supports the model's usefulness for any voltage waveform, and not simply those that are periodic.

4.3 Other predictable examples

While they are not validated in a lab during the physics practical course, two more examples using inductors and capacitors have predictable solutions. From the formulae, $V_L = L\frac{\mathrm{d}I}{\mathrm{d}t}$ and $I = C\frac{\mathrm{d}V_C}{\mathrm{d}t}$, it was predicted that a periodic stepping voltage in an inductor would produce a sawtooth current, and that the opposite would be true for a capacitor. By setting the resistance arbitrarily low, Figure 7 demonstrates the stepping voltage in an inductor, $L = 1[\text{H}]$, and a sawtooth voltage in a capacitor, $C = 1[\text{F}]$.

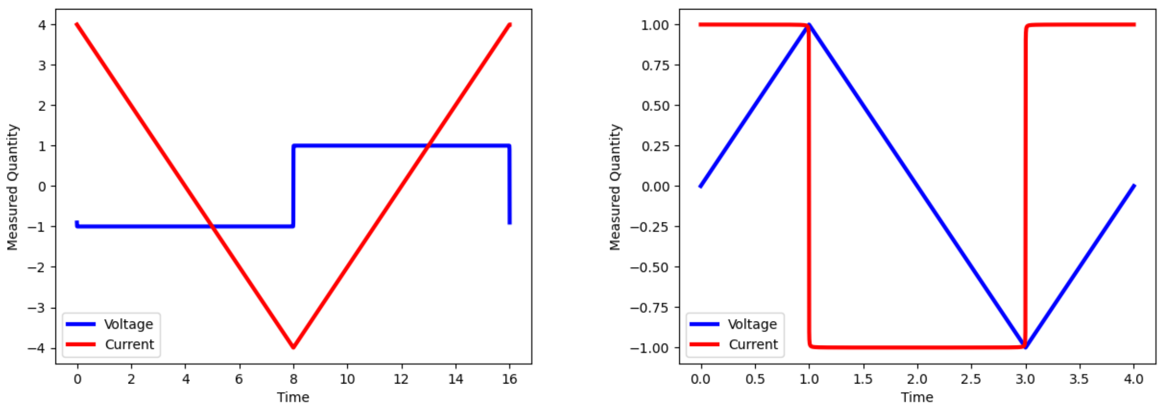

Figure 7: Inductor $L = 1[\text{H}]$ with stepping voltage (left); Capacitor $C = 1[\text{F}]$ with sawtooth voltage (right)

As expected, in both cases, the result follows what was predicted from the differential equations, as the derivative of a sawtooth wave is a stepping wave.

5 Interpretation and Extension

5.1 Accuracy of predictions

All of the tests so far would indicate that for high values of N and P, the waveforms produced by the model tend to the true waveforms. This is especially surprising for the cases where there were discontinuities in the current and its first derivative. In these cases, the Fourier series does not converge uniformly to the desired function (Lukas, 2023). However, with a high enough sampling rate and with enough terms in the series, these effects can be made less and less severe.

Given that this approximation has two computing parameters, N and P, it is important to also discuss which one should be prioritised for highest accuracy with lowest runtimes.

The number of samples, P, is equal to the number of terms in the sum for each integral coefficient approximation. The number of phasors, N, is equal to the number of integral approximations which are calculated. Assuming the time to calculate the Fourier coefficients for voltage and current is much larger than the time to create arrays and produce the plot, then the best estimate for the time complexity of the program is $O(NP)$ in Big O notation. This is subject, however, to the fact that N also signifies the number of times the impedance is calculated and applied. This means that increasing N does incur greater time losses than increasing P.

This result is acceptable when considering the importance of the role that each statistic plays. The value of P determines how accurately the voltage is represented when calculating the current. Following this, the value of N dictates how accurately the current is calculated, based on the approximated voltage waveform. As such, the value in increasing N is strictly limited by the value of P, and increasing N will only calculate the true current more accurately if P is sufficiently high.

5.2 Extension to bandpass filter

Having shown how effective the model is in predicting the current in a circuit, we extend its utility only slightly by considering other quantities determined with AC circuit analysis. Figure 8 shows a bandpass filter with input and output voltage.

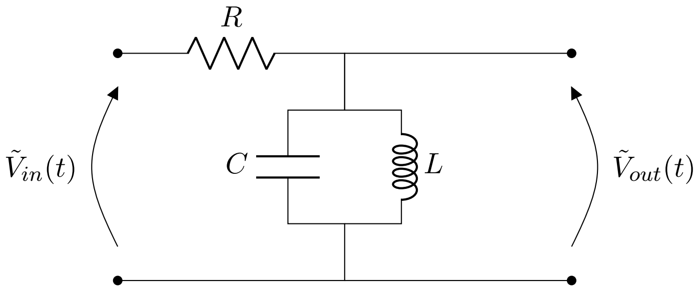

Figure 8: Passive bandpass filter circuit

Assuming a sinusoidal input voltage, such that $\tilde{V}_{in}(t) \propto e^{i\omega t}$, the circuit is treated as a potential divider, finding that $\frac{\tilde{V}_{out}(t)}{\tilde{V}_{in}(t)} = \left(1 + \frac{iR}{L\omega_0}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)^{-1}$ where $\omega_0 = (LC)^{-1/2}$. The purpose of the bandpass filter is to bias the output voltage so that signals with frequency ω_0 pass unaltered, while the level of attentuation increases with deviation from ω_0. The level of attentuation for the same deviation of frequency is increased by increasing the value of R.

This situation is very similar to the previous calculation, except we have replaced $\tilde{V}(t) \mapsto \tilde{V}_{in}(t)$, $\tilde{I}(t) \mapsto \tilde{V}_{out}(t)$ and $Z(\omega) \mapsto 1 + \frac{iR}{L\omega_0}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$. Thus, we attempt to solve for the output voltage for an arbitrary input voltage, analogously to before.

To test this, we chose values of $L = 0.1$[H], $C = 10$[F] and $R = 10000[\Omega]$. This sets $\omega_0 = 1[\mathrm{rad\,s^{-1}}]$ and ensures that only signals with this frequency will be allowed to pass. Figure 9 shows the passing of three different square waves into the filter, with angular frequencies of $1[\mathrm{rad\,s^{-1}}]$, $\frac{1}{2}[\mathrm{rad\,s^{-1}}]$ and $\frac{1}{3}[\mathrm{rad\,s^{-1}}]$.

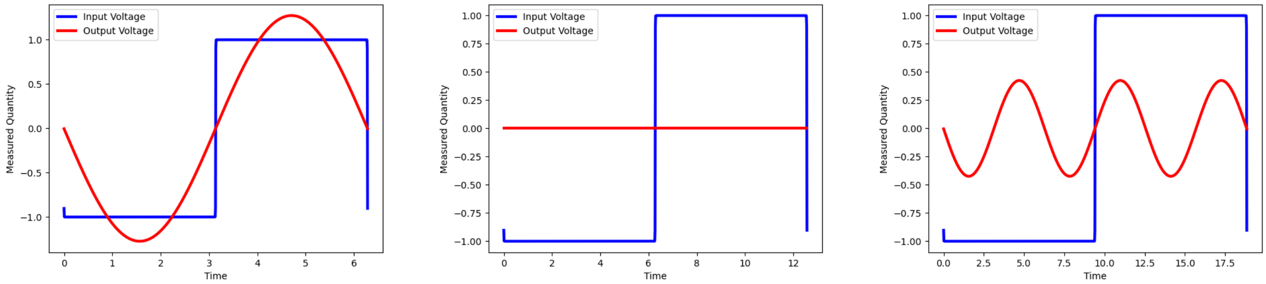

Figure 9: Bandpass filter $\omega_0 = 1[\mathrm{rad\,s^{-1}}]$ with square wave of $\omega/\omega_0 = 1$(left); $\frac{1}{2}$(middle); $\frac{1}{3}$(right)

Figure 9 demonstrates the correct theoretical results, as we expect it to illustrate the amplitude of each sinusoid term with angular frequency ω_0 in the Fourier series of a square wave of angular frequency ω. The result demonstrates how the sinusoid Fourier coefficient for a square wave will vanish unless ω_0/ω is an odd integer.

6 Conclusions

This experiment has validated an approach to analysing passive linear circuits using the complex Fourier series. The program reaps the benefits of both differential equation and complex AC impedance solutions to circuits by empowering the user to calculate the current response of any circuit to any arbitrary input voltage waveform. It completes this process to a high degree of accuracy with very low runtimes by calculating the Fourier coefficients for a piecewise-linear approximation to the voltage waveform, rather than performing integrals.

The only discovered flaw with the approach is that it encounters exceptions when the impedance for a certain frequency tends to infinity. This means the user must compute their own value for the complex impedance, limiting the usefulness of the tool by the user's knowledge.

The approach, however, has demonstrating capabilities beyond calculating currents from voltages, into the field of predicting any quantities in a circuit that could normally be attained by complex number methods, were the source sinusoidal. With these observations in mind, the program appears to be an effective simulator for passive linear circuits.

References

[1] A. Lukas. (2023). *Mathematical Methods*. Rudolf Peierls Centre for Theoretical Physics, University of Oxford.

[2] R. Smith. (2022). *1st-year Circuits: Lectures 3 to 4*. Department of Physics, University of Oxford.

[3] R. Smith. (2022). *1st-year Circuits: Lectures 5 to 7*. Department of Physics, University of Oxford.

[4] University of Oxford. (2022). *CO20: Open Simulation Topic*. Department of Physics, University of Oxford.

Appendix A The Complex Fourier Series

The following summary is adapted from (Lukas, 2023, p. 50).

Consider a vector space V over the field $\mathbb{C}$ spanned by the vector functions defined as:

$$|\epsilon_n\rangle \equiv \epsilon_n : \mathbb{R} \longrightarrow \mathbb{C} \quad \text{for} \quad n \in \mathbb{Z}$$
$$\text{such that} \quad t \mapsto \epsilon_n(t) = \frac{1}{\sqrt{T}}\, e^{i\omega_n t} \quad \{\omega_n = 2\pi n/T\}.$$

We first note that $\overline{\epsilon_n(t)} = \epsilon_{-n}(t)$ and by interpreting i as the imaginary unit and T as a real positive number, we see that, for $n \neq 0$, $|\epsilon_n\rangle$ is a phasor of length $1/\sqrt{T}$ that rotates (anti)clockwise for (positive)negative values of n with period T/n. Furthermore, we have that $|\epsilon_0\rangle$ is a constant of $1/\sqrt{T}$.

On V, we define vector addition and scalar multiplication as is standard for vector functions, allowing the formation of linear combinations of the form:

$$|\hat{f}\rangle = \sum_{n\in\mathbb{Z}} c_n |\epsilon_n\rangle \equiv \hat{f} = \sum_{n\in\mathbb{Z}} c_n \epsilon_n.$$

By integrating over period T, we can also define the sesquilinear form, $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$, such that:

$$\Big(|\hat{f}\rangle, |\hat{g}\rangle\Big) \mapsto \langle |\hat{f}\rangle, |\hat{g}\rangle \rangle \equiv \int_0^T \overline{\hat{f}(t)}\, \hat{g}(t)\, \mathrm{d}t.$$

This form satisfies all the conditions of a complex scalar product, including that $\overline{\langle |\hat{f}\rangle, |\hat{g}\rangle\rangle} = \langle |\hat{g}\rangle, |\hat{f}\rangle\rangle$. By checking that $\langle |\epsilon_m\rangle, |\epsilon_n\rangle\rangle = \delta_{mn}$, we find that the $|\epsilon_n\rangle$ form an orthonormal basis for V, with respect to this scalar product. This allows us to define the basis elements of V^*, the dual space to V:

$$\langle \epsilon_n| \equiv \epsilon_n^* : V \longrightarrow \mathbb{C} \quad \text{for} \quad n \in \mathbb{Z}$$
$$\text{such that} \quad |\hat{f}\rangle \mapsto \epsilon_n^*(|\hat{f}\rangle) = \int_0^T \epsilon_{-n}(t)\, \hat{f}(t)\, \mathrm{d}t \quad \text{for} \quad \hat{f} \in V.$$

We can also form linear combinations of these dual vectors. If $|\hat{f}\rangle = \sum_{n\in\mathbb{Z}} c_n |\epsilon_n\rangle$, then we confirm that $\langle \hat{f}| = \sum_{n\in\mathbb{Z}} \overline{c_n} \langle \epsilon_n|$. Using these dual vectors, we rewrite the notation for the scalar product between $|\hat{f}\rangle$ and $|\hat{g}\rangle$ as $\langle \hat{f}|\hat{g}\rangle$.

We now consider the vector space consisting of all periodic complex functions $\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ with period T such that $t \mapsto \hat{f}(t)$. Fourier's proof shows that this is exactly the space V. This makes it possible to express any complex-valued periodic function as a linear combination of the afforementioned complex exponentials with discrete angular frequencies.

To find the coordinates of the function with respect to this basis, we use the inner product found.

$$\text{If} \quad \hat{f}(t) = \sum_{n\in\mathbb{Z}} c_n \epsilon_n(t) \quad \text{then} \quad c_m = \langle \epsilon_m | \hat{f}\rangle = \int_0^T \epsilon_{-m}(t)\, \hat{f}(t)\, \mathrm{d}t \quad \text{for} \quad m \in \mathbb{Z}.$$

Appendix B Fourier Series of Piecewise-Linear Voltage Approximation

Given an arbitrary complex voltage waveform with period T of the form $\tilde{V} = \tilde{V}(t)$, we produce a piecewise linear approximation, with known Fourier series, thus allowing a computer to rapidly approximate the voltage fourier coefficients without performing any integrals.

The piecewise-linear function approximation

The user decides a positive integer number of samples P. This provides a set of P complex numbers, z_k, for $k \in \{0, 1, \ldots, P-1\}$, defined by the value of the voltage at equally-spaced times in T such that $z_k = \tilde{V}\left(\frac{k}{P}T\right)$. In essence, the piecewise approximation of $\tilde{V}$ is a function that connects each of these points, at their respective time, in straight line segments.

Immediately, we can calculate the gradients of these piecewise segments, m_k, where $m_k = \frac{P}{T}(z_{k+1} - z_k)$. This implies that both $z_k = m_k \frac{k}{P}T + c_k$ and $z_{k+1} = m_k \frac{k+1}{P}T + c_k$ for some set of complex intercepts, c_k, given by $c_k = z_k - m_k \frac{k}{P}T$.

Finally, this gives us a piecewise approximate complex voltage, $\tilde{V}_P : \mathbb{R} \longrightarrow \mathbb{C}$, such that $t \mapsto \tilde{V}_P(t)$ defined between $t \in [0, T]$ and periodic elsewhere such that:

$$\tilde{V}_P(t)\big|_{t\in\left[\frac{k}{P}T, \frac{k+1}{P}T\right)} = m_k t + c_k \quad \text{for} \quad k \in \{0, 1, \ldots, P-1\}.$$

The complex Fourier coefficients

We now apply the complex Fourier series to the piecewise function. If $\tilde{V}_P(t) = \sum_{n\in\mathbb{Z}} c_n \epsilon_n(t)$ for c_n, we must treat the $n = 0$ case separately.

When $n = 0$, the exponential term disappears and the inner product is calculated as:

$$\begin{aligned}
c_0 &= \int_0^T \epsilon_{-0}(t)\, \tilde{V}_P(t)\, \mathrm{d}t \\
&= \frac{1}{\sqrt{T}} \sum_{k=0}^{P-1} \int_{\frac{k}{P}T}^{\frac{k+1}{P}T} (m_k t + c_k)\, \mathrm{d}t \\
&= \frac{1}{\sqrt{T}} \sum_{k=0}^{P-1} \left[\frac{m_k}{2} t^2 + c_k t\right]_{\frac{k}{P}T}^{\frac{k+1}{P}T} \\
&= \frac{1}{\sqrt{T}} \sum_{k=0}^{P-1} \left(\frac{m_k}{2}\left(\left(\frac{k}{P}T\right)^2 + 2\frac{k}{P}T + 1\right) + c_k\left(\frac{k}{P}T + 1\right) - \frac{m_k}{2}\left(\frac{k}{P}T\right)^2 - c_k\frac{k}{P}T\right) \\
&= \frac{\sqrt{T}}{P} \sum_{k=0}^{P-1} \left(m_k \frac{k}{P}T + c_k + \frac{1}{2} m_k \frac{T}{P}\right) \\
&= \frac{\sqrt{T}}{P} \sum_{k=0}^{P-1} z_k + \cancel{\frac{T\sqrt{T}}{2P^2} \sum_{k=0}^{P-1} (z_{k+1} - z_k)} \quad \{z_0 = z_P\} \\
&= \frac{\sqrt{T}}{P} \sum_{k=0}^{P-1} z_k.
\end{aligned}$$

Noting that $\epsilon_0(t) = 1/\sqrt{T}$, this means that the constant term in the series is $\sum_{k=0}^{P-1} z_k / P$, which is a sort of "centre of mass" for the points.

Using a similar process, we calculate the coefficient where $n \neq 0$ to be:

$$
\begin{aligned}
c_n &= \int_0^T \epsilon_{-n}(t)\, \tilde{V}_P(t)\, \mathrm{d}t \\
&= \frac{1}{\sqrt{T}} \sum_{k=0}^{P-1} \int_{\frac{k}{P}T}^{\frac{k+1}{P}T} (m_k t + c_k)\, e^{-i\omega_n t}\, \mathrm{d}t \\
&= \frac{1}{{\omega_n}^2\sqrt{T}} \sum_{k=0}^{P-1} \left[e^{-i\omega_n t} \left(m_k + i\omega_n \left(m_k t + c_k \right) \right) \right]_{\frac{k}{P}T}^{\frac{k+1}{P}T} \\
&= \frac{1}{{\omega_n}^2\sqrt{T}} \sum_{k=0}^{P-1} m_k \left(e^{-i\omega_n \frac{k+1}{P}T} - e^{-i\omega_n \frac{k}{P}T} \right) \\
&\quad + \cancel{\frac{i}{\omega_n\sqrt{T}} \sum_{k=0}^{P-1} \left(e^{-i\omega_n \frac{k+1}{P}T} z_{k+1} - e^{-i\omega_n \frac{k}{P}T} z_k \right)} \quad \{ e^{-i\omega_n \frac{0}{P}T} = e^{-i\omega_n \frac{P}{P}T} \} \\
&= \frac{1}{{\omega_n}^2\sqrt{T}} \sum_{k=0}^{P-1} e^{-i\omega_n \frac{k}{P}T} (m_{k-1} - m_k) \quad \{\text{define } m_{-1} = m_{P-1}\}.
\end{aligned}
$$

With these calculations completed, we write an expression for the Nth Fourier series approximant, $\tilde{V}_{P,N}(t)$, to the piecewise complex function, $\tilde{V}_P(t)$, being:

$$
\begin{aligned}
\tilde{V}_{P,N}(t) &= \sum_{n=-N}^{N} c_n \epsilon_n(t) \\
&= \frac{1}{P} \sum_{k=0}^{P-1} z_k + \frac{1}{T} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{{\omega_n}^2} \left(\sum_{k=0}^{P-1} e^{-i\omega_n \frac{k}{P}T} (m_{k-1} - m_k) \right) e^{i\omega_n t}.
\end{aligned}
$$

We, finally, have that $\lim_{N\to\infty} \tilde{V}_{P,N}(t) = \tilde{V}_P(t)$ and $\lim_{P\to\infty} \tilde{V}_P(t) = \tilde{V}(t)$

This gives us the final form of the complex voltage, as a sum of AC components, for each a complex impedance can be applied based on the value of ω_n.

Appendix C Python Code

Find below the code which has taken a stepping voltage waveform, period of 100[s] period and underdamped LCR circuit impedance function, before plotting the current with the voltage.

```
# This script takes an arbitrary voltage waveform, as well as details for
# the complex AC impedance of a chosen circuit. By deconstructing the voltage
# waveform into AC components with a complex Fourier series, the current
# response is calculated and then plotted over 1 period with the voltage.

# Author: Ewan Thomas Beach
# Date: 23/11/2023
# Organisation: University of Oxford
# Department: Department of Physics


## Importing relevant packages

import matplotlib.pyplot as plt
import numpy as np
from numpy import pi,sin,cos,sqrt
from cmath import phase



### Section 1: User Inputs

## Waveform inputs

T = 100 # identify waveform period [+ve real]

def Vreal(t): # identify voltage waveform as function of time (starts t=0)
    if t<T/2:
        return 1
    else:
        return 0

def Vimag(t): # choose arbitrary imaginary voltage
    return 0


## Circuit components (LCR circuit as base)

c = 1 # include finite capacitor [boolean]
L = 1 # inductance of LCR circuit [+ve real or zero]
C = 0.5 # capacitance of LCR circuit [+ve real]
R = 0.2 # resistance of LCR circuit [+ve real]

def imp(W): # identify function for finding AC complex impedance of circuit
    if c == 1:
        return complex(R,W*L-1/W/C)
    else:
        return complex(R,W*L)
```

```
## Approximation and plotting preferences

P = 2500 # number of points in piecewise waveform approximation [+ve integer]
N = 2500 # maximum Fourier series partial sum index [+ve integer]
f = 1 # include voltage waveform on graph [boolean]



### Section 2: Piecewise Voltage Approximation

pt = np.full(P,T/P)
for i in range(P):
    pt[i]=i*pt[i] # defining array of times
px = np.array(list(map(Vreal,pt))) # defining array of voltage values
py = np.array(list(map(Vimag,pt))) # defining array of imaginary voltage values

v0 = complex(sum(px),sum(py))/P # immediately calculating DC voltage

pt = np.append(pt,T) # repeating initial term at end
px = np.append(px,px[0])
py = np.append(py,py[0])


## Calculating complex gradients

mx = np.zeros(P)
my = np.zeros(P)
for i in range(P):
    mx[i] = (px[i+1]-px[i])/(T/P)
    my[i] = (py[i+1]-py[i])/(T/P)
m = list(map(complex,mx,my))



### Section 3: Decomposing Voltage Signal

w = 2*pi*(np.array(range(2*N+1))-N)/T # assign frequencies

def complexp(phase):
    return complex(cos(phase),sin(phase))
expons = list(map(complexp,-w*T/P)) # produce list of complex exponentials

def vn(n): # function for finding complex voltage coefficient
    items = [0]*P
    for i in range(P):
        items[i] = expons[N+n]**i*(m[(i-1)%P]-m[i])
    return sum(items)/T/w[N+n]**2
```

```
## Collecting coefficients in a list

Vn = [0]*(2*N+1)
Vn[N] = v0
for i in range(N):
    Vn[i] = vn(i-N)
    Vn[2*N-i] = vn(N-i)


## Calculate voltage radii and initial phases

Rv = np.array(list(map(abs,Vn)))
Ov = np.array(list(map(phase,Vn)))



### Section 4: Calculating Current Waveform

In = Vn # mimicking voltage coefficients

if v0 != 0: # giving special (capacitor) treatment to the DC value
    if c == 1:
        In[N] = 0
    else:
        In[N] = In[N]/R


## Applying impedances to voltage coefficients

for i in range(N):
    In[i] = In[i]/imp(w[i])
    In[2*N-i] = In[2*N-i]/imp(w[2*N-i])


## Calculate current radii and initial phases

Ri = np.array(list(map(abs,In)))
Oi = np.array(list(map(phase,In)))



### Section 5: Producing The Final Data Set

## Defining functions for calculating AC contributions

def gen_V_dis(t):
    return Rv*cos(w*t+Ov)
def gen_I_dis(t):
    return Ri*cos(w*t+Oi)
```

```
## Calculating voltage and current using radii and phase

V=np.zeros(P+1)
I=np.zeros(P+1)

for i in range(P+1):
    V[i] = sum(gen_V_dis(pt[i]))
    I[i] = sum(gen_I_dis(pt[i]))



### Section 6: Producing The Plot

if f:
    plt.plot(pt,V,"b",linewidth=3,label="Voltage")
plt.plot(pt,I,"r",linewidth=3,label="Current")
plt.xlabel("Time")
plt.ylabel("Measured Quantity")
plt.legend()

plt.show()
```